A CONTINUOUS FUNCTION WITH NO UNILATERAL DERIVATIVES*

BY

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1. Introduction

A. S. Besicovitch† has given a construction of an even continuous function B for which he asserts the property $D_+B(x) < D^+B(x)$, $(-1 < x < 1)$, and, in consequence of the evenness of B, the companion property $D_-B(x) < D^-B(x)$, $(-1 < x < 1)$; that is, B has nowhere a unilateral derivative finite or infinite. Later E. D. Pepper‡ examined this same function. Some readers have found difficulty in following the reasoning employed by both these authors, and it may be that in some minds doubt as to the existence of such a function has been raised by the theorem of S. Saks§ to the effect that the "functions of Besicovitch" constitute a set of only *first* category in the space C of continuous functions.||

In the present paper we, like Besicovitch, associate with a function having dense intervals of constancy another such function. The method of association used here, however, is purely arithmetic and differs essentially from that of Besicovitch. The function F which it is our purpose to define and investigate is even and continuous on the open interval $(-1, 1)$ and has the property

$$\liminf_{\xi \to x+} \left| \frac{F(\xi) - F(x)}{\xi - x} \right| < \limsup_{\xi \to x+} \left| \frac{F(\xi) - F(x)}{\xi - x} \right| = \infty, \qquad -1 < x < 1,$$

which, incidentally, the function B does not possess.

We remark that it is our intention to order the subsequent material so as to facilitate the reading of the proof of Theorem 5.2. Should the reader wish to acquaint himself at once with the formal definition of F, he may do so by examining §2 and Definitions 3.1, 3.3, 4.2, and 5.1.

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† Besicovitch, *Discussion der stetigen Funktionen im Zusammenhang mit der Frage über ihre Differentiierbarkeit*, Bulletin de l'Académie des Sciences de Russie, vol. 19 (1925), pp. 527–540.

‡ Pepper, *On continuous functions without a derivative*, Fundamenta Mathematicae, vol. 12 (1928), pp. 244–253.

§ Saks, *On the functions of Besicovitch in the space of continuous functions*, Fundamenta Mathematicae, vol. 19 (1932), pp. 211–219.

|| It may be recalled that Banach, *Über die Baire'sche Kategorie gewisser Funktionenmengen*, Studia Mathematica, vol. 3 (1931), p. 174, has shown that in C the set of functions having at no point a *finite* unilateral derivative is of *second* category.

2. Notation and conventions

We employ the symbol $[x_1, x_2]$ to denote a closed interval, the symbol (x_1, x_2) to denote an open interval. The latter notation will be understood to imply that $x_1 < x_2$. By the word *set* we understand a set of real numbers, by the word *function* a function on a set to a set. The letters n, N, ν are permitted to assume integer values only. We denote the outer Lebesgue measure of a set A by $|A|$.

We shall find it convenient to refer to an open interval (α, β) as an *interval of* a set A if $(\alpha, \beta) \subset A$ with α and β elements of the closure of the complement of A. In case A is open this means α and β are not elements of A.

In connection with a function f we employ the notations:

$$K(f) = \text{the interior of the set } \underset{x}{E}\left[\lim_{h\to 0} [f(x+h) - f(x)]/h = 0\right],$$

$$H(f) = \text{the complement of } K(f) \text{ with respect to the domain of } f,$$

$$Z(f) = \underset{x}{E}[f(x) = 0], \qquad P(f) = \underset{x}{E}[f(x) > 0].$$

We also agree that if A is a set and $r \geqq 0$, then the set $A^{(r)}$ shall be defined as follows: $x \in A^{(r)}$ if $x = \pm 1$ or if there is an interval (a, b) such that $x \in (a, b) \subset A$ with $b - a > r$.

3. The sequence $\{\lambda_n\}$ and the function θ

3.1. Definition. *$\{\lambda_n\}$ is the sequence, defined on all integers n, for which*

$$\lambda_n = 1/2 + n/2(|n| + 3).$$

In addition to the fact that $\lambda_n \to 1$ or 0 according as $n \to +\infty$ or $-\infty$, the properties of $\{\lambda_n\}$ for which we shall have use are embodied in the following lemma:

3.2. Lemma. *For each integer n, $\lambda_n + \lambda_{-n} = 1$ and $0 < \lambda_{n+1} - \lambda_n < \lambda_n^2 < \lambda_n$.*

Proof. The first relation being evident, we prove the second. For each integer n, $|n|(n+1) - |n+1|n = 0$, and we have

$$\begin{aligned}\lambda_{n+1} - \lambda_n &= \frac{1}{2}\left(\frac{n+1}{|n+1|+3} - \frac{n}{|n|+3}\right)\\ &= \frac{1}{2}\left(\frac{|n|(n+1) + 3(n+1) - n|n+1| - 3n}{(|n+1|+3)(|n|+3)}\right)\\ &= \frac{3}{2(|n+1|+3)(|n|+3)} = \left(\frac{3}{2(|n|+3)}\right)^2 \cdot \frac{2(|n|+3)}{3(|n+1|+3)}\end{aligned}$$

$$\leqq \left(\frac{3+n+|n|}{2(|n|+3)}\right)^2 \cdot \frac{2(|n+1|+3+1)}{3(|n+1|+3)}$$

$$= (\lambda_n)^2 \cdot \frac{2}{3}\left(1+\frac{1}{|n+1|+3}\right)$$

$$\leqq (\lambda_n)^2(2/3)(1+1/3) = (\lambda_n)^2(8/9) < (\lambda_n)^2 < \lambda_n.$$

Since λ_n converges to 1 or 0 according as $n \to +\infty$ or $-\infty$, and since $0<\lambda_n<\lambda_{n+1}<1$ for each integer n (see 3.2), it follows that there is a unique function θ determined by the following definition:

3.3. Definition. *With* $\gamma_n=\lambda_n$ *for* n *odd and* $\gamma_n=(\lambda_n)^{1/2}$ *for* n *even and* $\mathfrak{H}$ *a particular non-dense closed set which enjoys the property**

$$|[\lambda_n - h, \lambda_n]\cdot\mathfrak{H}| > 0, \quad |[\lambda_n, \lambda_n + h]\cdot\mathfrak{H}| > 0, \qquad h > 0, -\infty < n < \infty,$$

θ *is the function on* $(0, 2)$ *defined by the following relations*:

$$\theta(x) = \gamma_n + (\gamma_{n+1} - \gamma_n)|[\lambda_n, x]\cdot\mathfrak{H}|/|[\lambda_n, \lambda_{n+1}]\cdot\mathfrak{H}|, \qquad \lambda_n \leqq x \leqq \lambda_{n+1},$$

$$\theta(1) = 1, \qquad \theta(x) = \theta(2-x), \qquad 0 < x < 2.$$

3.4. Theorem. *The function* θ *has the following properties*:

(a) θ *is continuous on* $(0, 2)$ *and*

$$\lim_{\xi\to 0+} \theta(\xi) = 0 < x/2 \leqq \theta(x) \leqq (x+3)/4 \leqq \theta(1) = 1, \qquad 0 < x < 1;$$

(b) $K(\theta)$† *is dense in* $(0, 2)$, *while* $\lambda_n \in H(\theta)$ *for each integer* n;

(c) $$\lim_{\xi\to x+} |[\theta(\xi) - \theta(x)]/(\xi - x)| < \infty, \qquad 0 < x < 2;$$

(d) *if* (α, β) *is an interval of* $K(\theta)$, *then* $\theta[(\alpha+\beta)/2] \geqq (\beta-\alpha)^{1/2}$.

Proof. In order to establish these properties we note first that for each integer n, θ is clearly continuous on $[\lambda_n, \lambda_{n+1}]$ and that

$$\text{(1)} \qquad \lambda_n \leqq \theta(x) \leqq (\lambda_{n+1})^{1/2}, \qquad \lambda_n \leqq x \leqq \lambda_{n+1}.$$

Thus from (1), the symmetry of θ, and the equalities

$$\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} (\lambda_n)^{1/2} = 1, \qquad \lim_{n\to-\infty} \lambda_n = \lim_{n\to-\infty} (\lambda_n)^{1/2} = 0,$$

it follows that θ is continuous on $(0, 2)$ with $\theta(\xi)\to 0$ as $\xi\to 0+$. Observing that the relations

$$\lambda_{n+1} - \lambda_n < \lambda_n, \qquad \lambda_n + \lambda_{-n} = 1, \qquad n \text{ an integer},$$

* This is an almost immediate consequence of the existence of non-dense closed sets of positive measure.

† The meanings of $K(\theta)$, $H(\theta)$, and an *interval of* a set are explained in §2.

found in Lemma 3.2 imply the relations

$$(2) \qquad \lambda_{n+1} < 2\lambda_n, \qquad \lambda_n = 1 - \lambda_{-n}, \qquad n \text{ an integer},$$

and recalling that the geometric mean never exceeds the arithmetic, we conclude from (1) that $\lambda_n \leqq x \leqq \lambda_{n+1}$ implies

$$\begin{aligned} 0 < x/2 \leqq \lambda_{n+1}/2 < \lambda_n &\leqq \theta(x) \leqq (\lambda_{n+1})^{1/2} = (1 - \lambda_{-n-1})^{1/2} \leqq (1 - \lambda_{-n}/2)^{1/2} \\ &= \left(\frac{1 + \lambda_n}{2}\right)^{1/2} = \left[1 \cdot \left(\frac{1 + \lambda_n}{2}\right)\right]^{1/2} \leqq \frac{1}{2}\left(1 + \frac{1 + \lambda_n}{2}\right) \\ &= \frac{3 + \lambda_n}{4} \leqq \frac{3 + x}{4}; \end{aligned}$$

whence (a) is established.

Using the notation of 3.3 for the remainder of the proof, and deferring consideration of (b) until last, we begin checking (c) by noting that (1) and (2) imply

$$\begin{aligned} |\theta(1) - \theta(\xi)| &= 1 - \theta(\xi) \leqq 1 - \lambda_n = \lambda_{-n} \leqq 2\lambda_{-n-1} \\ &= 2(1 - \lambda_{n+1}) \leqq 2(1 - \xi), \qquad \lambda_n \leqq \xi \leqq \lambda_{n+1}. \end{aligned}$$

From this and the symmetry of θ it follows that for $x = 1$

$$\lim_{\xi \to x} |[\theta(\xi) - \theta(x)]/(\xi - x)| < \infty;$$

for other $x \varepsilon (0, 2)$, this inequality results from the relation (see 3.3 and note that $|\gamma_{n+1} - \gamma_n| < 1$ for each integer n)

$$|\theta(x_2) - \theta(x_1)| \leqq |x_2 - x_1| / |[\lambda_n, \lambda_{n+1}] \cdot \mathfrak{H}|, \qquad \lambda_n \leqq x_1, x_2 \leqq \lambda_{n+1}.$$

In order to check (d), we prove first that $\lambda_n \varepsilon H(\theta)$ for each integer n. If n is odd, then $\gamma_n = \lambda_n < (\lambda_{n+1})^{1/2} = \gamma_{n+1}$ and $\gamma_{n+1} - \gamma_n > 0$, so that the combination of this last relation with the property of $\mathfrak{H}$ stipulated in 3.3 yields the relations $\lambda_n \varepsilon H(\theta)$, $\lambda_{n+1} \varepsilon H(\theta)$; whence $\lambda_n \varepsilon H(\theta)$ for each integer n. This of course implies $1 \varepsilon H(\theta)$, so that upon letting (α, β) be any interval of $K(\theta)$ we have either $(\alpha, \beta) \subset (0, 1)$ or $(\alpha, \beta) \subset (1, 2)$. Thus in order to verify

$$\theta([\alpha + \beta]/2) \geqq (\beta - \alpha)^{1/2}$$

there is, in view of the symmetry of θ, no loss in generality in assuming $(\alpha, \beta) \subset (0, 1)$. Since $\lambda_n \varepsilon H(\theta)$ for each integer n, it is clear that an integer N exists for which $\lambda_N \leqq \alpha < \beta \leqq \lambda_{N+1}$, and hence from (1) and Lemma 4.2 follows the relation

$$\theta([\alpha + \beta]/2) \geqq \lambda_N > (\lambda_{N+1} - \lambda_N)^{1/2} \geqq (\beta - \alpha)^{1/2}.$$

The property (d) is now established as well as the second part of (b); the first part of (b) is a consequence of the fact that the complement of $\mathfrak{H}$ is an everywhere dense open set.

4. Functions of class $\mathfrak{A}$; the fundamental operation

4.1. Definition. *A function f on $(-1, 1)$ is said to belong to the class $\mathfrak{A}$ if it possesses the following properties:*

(a) *f is continuous with $0 \leqq f(-x) = f(x)$ for $x \in (-1, 1)$;*

(b) *$K(f)$ and $P(f)$ are dense in $(-1, 1)$;*

(c) *Corresponding to each interval (α, β) of the open set $P(f)$ there exists a number $h \geqq ([\beta-\alpha]/2)^{1/2}$ such that $x \in (\alpha, \beta)$ implies*

$$f(x) = h \cdot \theta[(2x - 2\alpha)/(\beta - \alpha)].$$

In the next definition we exhibit the operation which is fundamental in the present development and which, among other things, associates with each function in $\mathfrak{A}$ another such function.

4.2. Definition. *With each function f on $(-1, 1)$ we associate the function $\bar{f}$, likewise on $(-1, 1)$, for which $x \in H(f)$ implies $\bar{f}(x) = 0$, and $x \in K(f)$ implies $\bar{f}(x) = h \cdot \theta[(2x - 2\alpha)/(\beta - \alpha)]$, where (α, β) is the interval of $K(f)$ of which x is an element and h is the lesser of the numbers**

$$\left(\inf \left\{K(f)^{(\beta-\alpha)} \cdot \left[\frac{\alpha+\beta}{2}, 1\right]\right\} - \sup \left\{K(f)^{(\beta-\alpha)} \cdot \left[-1, \frac{\alpha+\beta}{2}\right]\right\}\right)^{1/2}$$

and

$$\frac{|f[(\alpha + \beta)/2]|}{2^{1/2}}.$$

We devote 4.3–4.9 to the study of this operation.

4.3. Lemma. *If $f \in \mathfrak{A}$ and (α, β) is an interval of $K(f)$ with $(\alpha+\beta)/2 = \mu$, then $f(\mu) \geqq (\beta - \alpha)^{1/2}$.*

Proof. The satisfaction of 4.1(b) by f implies $K(f) \subset P(f)$ which in turn implies the existence of an interval (α', β') of $P(f)$ for which $\alpha' < \alpha < \beta < \beta'$. Now let

$$\mu' = \frac{\alpha' + \beta'}{2}, \qquad \alpha'' = \frac{2\alpha - 2\alpha'}{\beta' - \alpha'}, \qquad \beta'' = \frac{2\beta - 2\alpha'}{\beta' - \alpha'}, \qquad \mu'' = \frac{\alpha'' + \beta''}{2},$$

and observe that the relation (see 4.1(c))

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* By $K(f)^{(\beta-\alpha)}$ we understand $[K(f)]^{(\beta-\alpha)}$. The latter notation is explained in §2.

$$f(x) = f(\mu')\theta[(2x - 2\alpha')/(\beta' - \alpha')], \qquad \alpha' < x < \beta',$$

implies (α'', β'') is an interval of $K(\theta)$. Making use of 3.4(d), 4.1(c), and the above relations we conclude

$$\begin{aligned} f(\mu) &= f(\mu')\theta[(2\mu - 2\alpha')/(\beta' - \alpha')] = f(\mu')\theta(\mu'') \geqq f(\mu')(\beta'' - \alpha'')^{1/2} \\ &\geqq \{[(\beta' - \alpha')/2]\cdot(\beta'' - \alpha'')\}^{1/2} \\ &= \{[(\beta' - \alpha')/2]\cdot 2(\beta - \alpha)/(\beta' - \alpha')\}^{1/2} = (\beta - \alpha)^{1/2}. \end{aligned}$$

4.4. LEMMA. *If $f \in \mathfrak{A}$ and (α, β) is an interval of $K(f)$ with $(\alpha+\beta)/2 = \mu$, then $\bar{f}(\mu) \geqq [(\beta-\alpha)/2]^{1/2}$ and $x \in (\alpha, \beta)$ implies*

$$0 < \bar{f}(-x) = \bar{f}(x) = \bar{f}(\mu)\theta[(2x - 2\alpha)/(\beta - \alpha)] \leqq \bar{f}(\mu) \leqq f(x)/2^{1/2}.$$

Proof. Letting

$$\alpha_0 = \sup\{K(f)^{(\beta-\alpha)}\cdot[-1, \mu]\}, \beta_0 = \inf\{K(f)^{(\beta-\alpha)}\cdot[\mu, 1]\}$$

so that $\bar{f}(\mu)$ is the lesser of the numbers $f(\mu)/2^{1/2}$ and $(\beta_0 - \alpha_0)^{1/2}$, we note that the evenness of f implies $(-\beta, -\alpha)$ is an interval of $K(f)$ and implies the equivalence of the relations $x \in K(f)^{(\beta-\alpha)}$, $-x \in K(f)^{(\beta-\alpha)}$ which in turn imply

$$-\beta_0 = \sup\{K(f)^{(\beta-\alpha)}\cdot[-1, -\mu]\}, \quad -\alpha_0 = \inf\{K(f)^{(\beta-\alpha)}\cdot[-\mu, 1]\}.$$

Consequently, in view of the relation $f(\mu) = f(-\mu)$, it follows that $\bar{f}(\mu) = \bar{f}(-\mu)$, which in conjunction with 4.2 and the symmetry of θ assures us that $\bar{f}(x) = \bar{f}(-x)$ for $x \in (\alpha, \beta)$. Furthermore the relations

$$(\beta_0 - \alpha_0)^{1/2} \geqq (\beta - \alpha)^{1/2} > [(\beta - \alpha)/2]^{1/2}, \quad f(\mu)/2^{1/2} \geqq [(\beta - \alpha)/2]^{1/2},$$

the latter of which is a consequence of 4.3, entail

$$\bar{f}(\mu) \geqq [(\beta - \alpha)/2]^{1/2} > 0.$$

From this last relation, 4.2, and 3.4(a) it follows that $x \in (\alpha, \beta)$ implies

$$0 < \bar{f}(x) = \bar{f}(\mu)\cdot\theta[(2x - 2\alpha)/(\beta - \alpha)] \leqq \bar{f}(\mu) \leqq f(\mu)/2^{1/2} = f(x)/2^{1/2},$$

and the proof of the lemma is complete.

4.5. LEMMA. *If $f \in \mathfrak{A}$ then*

$$0 \leqq \bar{f}(-x) = \bar{f}(x) \leqq f(x)/2^{1/2}, \qquad -1 < x < 1.$$

Proof. Since f is even, $x \in H(f)$ implies $-x \in H(f)$. Hence from 4.2 we conclude that for $x \in H(f)$

$$0 = \bar{f}(-x) = \bar{f}(x) \leqq f(x)/2^{1/2},$$

and the preceding lemma completes the proof.

4.6. LEMMA. *If* $f \in \mathfrak{A}$ *then* $K(f) = P(\bar{f})$, $H(f) = Z(\bar{f})$.

Proof. Lemma 4.4 implies $K(f) \subset P(\bar{f})$. Definition 4.2 implies $H(f) \subset Z(\bar{f})$. Lemma 4.5 implies that $P(\bar{f})$ and $Z(\bar{f})$ are complementary in $(-1, 1)$. The proof is complete.

4.7. LEMMA. *If* $f \in \mathfrak{A}$ *and* (α, β) *is an interval of* $K(f)$, *then* α *and* $(\alpha+\beta)/2$ *are elements of* $H(\bar{f})$ *and we have*

$$\bar{f}(\alpha) = \lim_{\xi\to\alpha+} \bar{f}(\xi) = 0 \leqq \liminf_{\substack{\xi\to\alpha+\\ \xi\in H(\bar{f})}} \frac{\bar{f}(\xi) - \bar{f}(\alpha)}{\xi - \alpha} < \limsup_{\substack{\xi\to\alpha+\\ \xi\in H(\bar{f})}} \frac{\bar{f}(\xi) - \bar{f}(\alpha)}{\xi - \alpha} = \infty.$$

Proof. Referring to 4.4 we note

$$\bar{f}(\xi) = h\theta[(2\xi - 2\alpha)/(\beta - \alpha)], \qquad \alpha < \xi < \beta,$$

where $h = \bar{f}[(\alpha+\beta)/2] > 0$; so that upon noting $\alpha \in H(f)$ and referring to 3.4(a), 4.2 we conclude that $\bar{f}(\xi) \to \bar{f}(\alpha) = 0$ as $\xi \to \alpha+$. Also upon defining $\xi_n = \alpha + \lambda_n(\beta - \alpha)/2$ and recalling that $\lambda_n \in H(\theta)$ for each integer n we conclude that $\xi_n \in H(\bar{f})$ for each integer n and (see 3.3)

$$\frac{\bar{f}(\xi_n)}{\xi_n - \alpha} = \frac{h\theta(\lambda_n)}{\lambda_n}\cdot\frac{2}{\beta - \alpha} = \frac{\lambda_n}{\lambda_n}\cdot\frac{2h}{\beta - \alpha} \to \frac{2h}{\beta - \alpha}, \qquad n \text{ odd}, n \to -\infty,$$

$$\frac{\bar{f}(\xi_n)}{\xi_n - \alpha} = \frac{h\theta(\lambda_n)}{\lambda_n}\cdot\frac{2}{\beta - \alpha} = \frac{(\lambda_n)^{1/2}}{\lambda_n}\cdot\frac{2h}{\beta - \alpha} \to \infty, \qquad n \text{ even}, n \to -\infty.$$

Since $\xi_n \to \alpha$ or $(\alpha+\beta)/2$ according as $n \to -\infty$ or $+\infty$, it is clear that α and $(\alpha+\beta)/2$ are elements of $H(\bar{f})$ and that the proof of the lemma is complete.

4.8. THEOREM. *If* $f \in \mathfrak{A}$ *then* $x \in H(f)\cdot P(f)$ *implies*

$$\lim_{\xi\to x+} \bar{f}(\xi) = \bar{f}(x) = 0 \leqq \liminf_{\substack{\xi\to x+\\ \xi\in H(\bar{f})}} \frac{\bar{f}(\xi) - \bar{f}(x)}{\xi - x} < \limsup_{\substack{\xi\to x+\\ \xi\in H(\bar{f})}} \frac{\bar{f}(\xi) - \bar{f}(x)}{\xi - x} = \infty.$$

Proof. In view of the preceding lemma we confine ourselves to the case in which x not only is an element of $H(f)\cdot P(f)$ but also a cluster point of $[x, 1]\cdot H(f)$.

Let $M(r) = \sup_{-1<t<1} \{\inf K(f)^{(r)}\cdot[t, 1] - \sup K(f)^{(r)}\cdot[-1, t]\}$, and note that the openness of $K(f)$ implies $(-1, 1)\cdot K(f)^{(r)} \uparrow K(f)$ as $r \to 0+$ so that the openness and denseness of $K(f)$ in $(-1, 1)$ imply* $M(r) \to 0$ as $r \to 0+$. Thus if $\{\xi_n\}$ is a sequence of points of $K(f)$ tending to x from the right and if the

* Let $\epsilon > 0$; let $-1 = a_0 < a_1 < \cdots < a_N = 1$ with $a_\nu - a_{\nu-1} < \epsilon/2$ for $\nu = 1, 2, \cdots, N$. Since there exists a positive number δ such that $K(f)^{(\delta)}\cdot[a_{\nu-1}, a_\nu]$ is non-void for $\nu = 1, 2, \cdots, N$, we conclude that $0 < r < \delta$ implies $M(r) < \epsilon$.

interval of $K(f)$ of which ξ_n is an element is denoted by (α_n, β_n), it becomes clear in the light of 4.5, 4.4, and 4.2 that

$$0 = \bar{f}(x) \leqq \bar{f}(\xi_n) \leqq \bar{f}[(\alpha_n + \beta_n)/2] \leqq [M(\beta_n - \alpha_n)]^{1/2} \to 0 \text{ as } n \to \infty.$$

Since $\bar{f}(x) = \bar{f}(\xi) = 0$ for $\xi \in H(f)$, it is now evident that $\bar{f}(\xi) \to \bar{f}(x) = 0$ as $\xi \to x+$. Now a sequence $\{(a_n, b_n)\}$ of intervals of $K(f)$ with $m_n = (a_n + b_n)/2$ exists such that $m_n \to x+$ and such that for each positive integer n the set $[x, m_n] \cdot K(f)^{(b_n - a_n)}$ is void and* the number $[2M(b_n - a_n)]^{1/2}$ is less than $f(m_n)$. Recalling 4.2 we have

$$\bar{f}(m_n) \leqq [M(b_n - a_n)]^{1/2} < f(m_n)/2^{1/2}, \quad n = 1, 2, 3, \cdots,$$

which with the fact that $[x, m_n] \cdot K(f)^{(b_n - a_n)}$ is void for $n = 1, 2, 3, \cdots$ implies

$$\begin{aligned} \bar{f}(m_n) &= \{\inf [m_n, 1] \cdot K(f)^{(b_n - a_n)} - \sup [-1, m_n] \cdot K(f)^{(b_n - a_n)}\}^{1/2} \\ &\geqq (m_n - x)^{1/2}, \qquad n = 1, 2, 3, \cdots. \end{aligned}$$

The Theorem is now a consequence of 4.5, the first conclusion of 4.7, and the following two relations which are valid for $n = 1, 2, 3, \cdots$:

$$\bar{f}(m_n)/(m_n - x) \geqq 1/(m_n - x)^{1/2}, \qquad \bar{f}(a_n) = \bar{f}(x) = 0.$$

4.9. THEOREM. *If $f \in \mathfrak{A}$ then $\bar{f} \in \mathfrak{A}$.*

Proof. That $\bar{f}$ is continuous on the right at each point of $H(f) \cdot P(f)$ is a consequence of 4.8; that the same is true on $H(f) \cdot Z(f)$ and on $K(f)$ is a consequence of 4.5 in the first case, and of 4.4, 3.4(a) in the second; that $\bar{f}$ satisfies 4.1(a) is now a consequence of 4.5.

Since $K(\theta)$ is dense in (0, 2) it follows from 4.4 that $K(\bar{f})$ is dense in $K(f)$ which is dense in $(-1, 1)$. Lemma 4.6 implies $K(f) = P(\bar{f})$ which of course assures us that $P(\bar{f})$ is dense in $(-1, 1)$ and that $\bar{f}$ exhibits property 4.1(b).

Lemma 4.4 combines with 4.6 to yield the satisfaction by $\bar{f}$ of 4.1(c): The proof is now complete.

5. THE FUNCTION F

5.1. DEFINITION. *$\{F_n\}$ is the sequence of functions on $(-1, 1)$ determined by*

$$F_0(x) = \theta(x + 1) \text{ for } -1 < x < 1, \qquad F_{n+1} = \bar{F}_n \text{ for } n = 0, 1, 2, \cdots.$$

F is the function defined by the relation

$$F = \sum_{n=0}^{\infty} (-1)^n F_n.$$

* Here we use the continuity of f, the inequality $f(x) > 0$, and the fact that $M(r) \to 0$ as $r \to 0+$.

We are now prepared to establish the following theorem:

5.2. THEOREM. *F is even and continuous on* $(-1, 1)$ *with*

$$\liminf_{\xi\to x+}\left|\frac{F(\xi)-F(x)}{\xi-x}\right| < \limsup_{\xi\to x+}\left|\frac{F(\xi)-F(x)}{\xi-x}\right| = \infty, \qquad -1<x<1,$$

*and, in particular,**

$$\liminf_{\xi\to x+}\left|\frac{F(\xi)-F(x)}{\xi-x}\right| = 0, \qquad x \in \prod_{\nu=0}^{\infty} P(F_\nu).$$

Proof. Denoting $\sum_{\nu=0}^{n}(-1)^\nu F_\nu$ by S_n, $F-S_n$ by R_n, $P(F_n)$ by $\mathfrak{P}_n$, $Z(F_n)$ by $\mathfrak{Z}_n$, $\prod_{\nu=0}^{\infty}\mathfrak{P}_\nu$ by $\mathfrak{P}$, $\sum_{\nu=0}^{\infty}\mathfrak{Z}_\nu$ by $\mathfrak{Z}$, and checking that $F_0 \in \mathfrak{A}$, we deduce from 4.9 that $n=0, 1, 2, \cdots$ implies $F_n \in \mathfrak{A}$ and hence that F_n is even and continuous. That F is even is now obvious; that it is continuous is a consequence of the easily verified relations (see 4.5)

(1) $$|R_n(x)| \leqq F_{n+1}(x) \leqq F_n(x)/2^{1/2} \leqq F_0(x)/2^{(n+1)/2} \leqq 2^{-(n+1)/2},$$
$$-1<x<1;\ n=0, 1, 2, \cdots.$$

A consequence of (1) of which considerable implicit use will be made is the fact that $F_{n+1}(x)=0$, or $x \in \mathfrak{Z}_{n+1}$, implies $R_n(x)=0$. It also follows from (1) that $\mathfrak{P}_n$ and $\mathfrak{Z}_n$ are complementary sets in $(-1, 1)$ for $n=0, 1, 2, \cdots$, that $\mathfrak{P}$ and $\mathfrak{Z}$ are likewise complementary in $(-1, 1)$, and that $\mathfrak{Z}_0 \subset \mathfrak{Z}_1 \subset \cdots \subset \mathfrak{Z}_m \subset \cdots$ with $(-1, 1)=\mathfrak{P}_0 \supset \mathfrak{P}_1 \supset \cdots \supset \mathfrak{P}_m \supset \cdots$.

We divide the remainder of the proof into three parts.

Part I. *If* $x \in \mathfrak{Z}$ *then*

$$\liminf_{\xi\to x+}\left|\frac{F(\xi)-F(x)}{\xi-x}\right| < \limsup_{\xi\to x+}\left|\frac{F(\xi)-F(x)}{\xi-x}\right| = \infty.$$

Letting x_0 be any element of $\mathfrak{Z}$ we introduce the notation

$$\Delta f(\xi) = [f(\xi) - f(x_0)]/(\xi - x_0)$$

to facilitate verification of $\liminf_{\xi\to x_0+}|\Delta F(\xi)| < \limsup_{\xi\to x_0+}|\Delta F(\xi)| = \infty$. To this end let N $(\geqq 1)$ be the integer for which $x_0 \in \mathfrak{Z}_N\cdot\mathfrak{P}_{N-1}$. Then since $x_0 \in \mathfrak{P}_n$ for $0\leqq n\leqq N-1$, it follows from 4.1(c) and 3.4(c) that for $0\leqq n\leqq N-1$, $\limsup_{\xi\to x_0+}|\Delta F_n(\xi)|<\infty$ and hence $\limsup_{\xi\to x_0+}|\Delta S_{N-1}(\xi)|<\infty$. On the other hand $\Delta R_N(\xi)=0$ for $\xi \in \mathfrak{Z}_{N+1}$. Moreover, in view of the relations $F_{N-1} \in \mathfrak{A}$, $F_N=\overline{F}_{N-1}$, and (see 4.6)

$$H(F_n) = Z(F_{n+1}), \qquad n = 0, 1, 2, \cdots,$$

* The set $\prod_{\nu=0}^{\infty}P(F_\nu)$ is a residual set of measure zero. The construction can be modified so that this set has measure arbitrarily close to two.

it may be seen that use of 4.8 yields

$$0 \leqq \liminf_{\substack{\xi \to x_0+ \\ \xi \in \mathfrak{Z}_{N+1}}} \Delta F_N(\xi) < \limsup_{\substack{\xi \to x_0+ \\ \xi \in \mathfrak{Z}_{N+1}}} \Delta F_N(\xi) = \infty .$$

Thus from the above follow the two relations

$$\begin{aligned} \liminf_{\xi \to x_0+} \left| \Delta F(\xi) \right| &\leqq \liminf_{\substack{\xi \to x_0+ \\ \xi \in \mathfrak{Z}_{N+1}}} \left| \Delta F(\xi) \right| = \liminf_{\substack{\xi \to x_0+ \\ \xi \in \mathfrak{Z}_{N+1}}} \left| \Delta S_{N-1}(\xi) + \Delta F_N(\xi) \right| \\ &\leqq \limsup_{\substack{\xi \to x_0+ \\ \xi \in \mathfrak{Z}_{N+1}}} \left| \Delta S_{N-1}(\xi) \right| + \liminf_{\substack{\xi \to x_0+ \\ \xi \in \mathfrak{Z}_{N+1}}} \left| \Delta F_N(\xi) \right| < \infty \end{aligned}$$

and

$$\begin{aligned} \limsup_{\xi \to x_0+} \left| \Delta F(\xi) \right| &\geqq \limsup_{\substack{\xi \to x_0+ \\ \xi \in \mathfrak{Z}_{N+1}}} \left| \Delta S_{N-1}(\xi) + \Delta F_N(\xi) \right| \\ &\geqq \limsup_{\substack{\xi \to x_0+ \\ \xi \in \mathfrak{Z}_{N+1}}} \left| \Delta F_N(\xi) \right| - \limsup_{\substack{\xi \to x_0+ \\ \xi \in \mathfrak{Z}_{N+1}}} \left| \Delta S_{N-1}(\xi) \right| = \infty , \end{aligned}$$

which in conjunction with the fact that x_0 is any point of $\mathfrak{Z}$ complete the proof of Part I.

Part II. *If* $x \in \mathfrak{P}$, *then*

$$\liminf_{\xi \to x+} \left| [F(\xi) - F(x)]/(\xi - x) \right| = 0.$$

Let x_0 be any point of $\mathfrak{P}$, let (α_n, β_n) be the interval of $\mathfrak{P}_n$ of which x_0 is an element, let $\mu_n = (\alpha_n + \beta_n)/2$. From 4.6 follows

$$(\alpha_n, \beta_n) \subset \mathfrak{P}_n \subset \mathfrak{P}_{\nu+1} = P(F_{\nu+1}) = K(F_\nu), \qquad 0 \leqq \nu \leqq n-1,$$

which implies

$$\begin{aligned} F_\nu(\mu_n) - F_\nu(x_0) &= F_\nu(\beta_n) - F_\nu(x_0) = 0, \\ S_\nu(\mu_n) - S_\nu(x_0) &= S_\nu(\beta_n) - S_\nu(x_0) = 0, \qquad 0 \leqq \nu \leqq n-1. \end{aligned} \tag{2}$$

Thus for n even and positive (note that $\beta_n \in \mathfrak{Z}_n$, and see inequality (1))

$$\begin{aligned} F(\beta_n) - F(x_0) &= S_{n-1}(\beta_n) - S_{n-1}(x_0) + F_n(\beta_n) - F_n(x_0) + R_n(\beta_n) - R_n(x_0) \\ &= -F_n(x_0) - R_n(x_0) \leqq -F_n(x_0) + \left| R_n(x_0) \right| \\ &\leqq -F_n(x_0) + F_n(x_0)/2^{1/2} = (-1 + 2^{-1/2})F_n(x_0) \leqq 0, \end{aligned} \tag{3}$$

whereas for n odd and positive

$$F(\beta_n) - F(x_0) = F_n(x_0) - R_n(x_0) \geqq F_n(x_0) - \left| R_n(x_0) \right| \geqq F_n(x_0)(1 - 2^{-1/2}) \geqq 0.$$

These last two relations coupled with the continuity of F imply the existence of $\{\xi_n\}$ for which

$$F(\xi_n) - F(x_0) = 0, \qquad \beta_{n+1} \leqq \xi_n \leqq \beta_n, \qquad n = 1, 2, 3, \cdots .$$

Hence in view of the relation (valid for $n=1, 2, 3, \cdots$)

$$(4) \qquad 0 < \xi_n - x_0 \leqq \beta_n - x_0 \leqq \beta_n - \alpha_n \leqq 2[F_n(\mu_n)]^2 \leqq 2^{-n+1} \to 0 \text{ as } n \to \infty$$

which is a consequence of 4.1(c), the equality $\theta(1)=1$, and inequality (1), the remainder of the proof of Part II is evident.

Part III. *If* $x \in \mathfrak{P}$ *then*

$$\limsup_{\xi \to x+} \left| [F(\xi) - F(x)]/(\xi - x) \right| = \infty .$$

Let x_0 be any element of $\mathfrak{P}$, and let α_n, β_n, and μ_n be as defined in Part II. By reasoning to be given later we have, in case $\alpha_n < x_0 < \mu_n$ with n even and positive,

$$(5) \qquad \begin{aligned} |F(\mu_n) - F(x_0)| &= |F_n(\mu_n) - F_n(x_0) - R_n(x_0)| \geqq F_n(\mu_n) - F_n(x_0) - R_n(x_0) \\ &\geqq F_n(\mu_n) - F_n(x_0) = F_n(\mu_n)\left\{1 - \theta\left(\frac{2x_0 - 2\alpha_n}{\beta_n - \alpha_n}\right)\right\} \\ &\geqq F_n(\mu_n)\left\{1 - \frac{1}{4}\left(\frac{2x_0 - 2\alpha_n}{\beta_n - \alpha_n} + 3\right)\right\} = F_n(\mu_n)\cdot\left(\frac{\mu_n - x_0}{2\beta_n - 2\alpha_n}\right) \\ &\geqq \left(\frac{\beta_n - \alpha_n}{2}\right)^{1/2}\cdot\left(\frac{\mu_n - x_0}{2\beta_n - 2\alpha_n}\right) = \frac{\mu_n - x_0}{2^{3/2}(\beta_n - \alpha_n)^{1/2}} \\ &\geqq 2^{(n-4)/2}\cdot(\mu_n - x_0) \geqq 2^{(n-6)/2}(\mu_n - x_0), \end{aligned}$$

whereas in case $\mu_n \leqq x_0 < \beta_n$ with n even and positive we have

$$(6) \qquad \begin{aligned} |F(\beta_n) - F(x_0)| &= |F_n(x_0) + R_n(x_0)| \geqq F_n(x_0) - |R_n(x_0)| \\ &\geqq F_n(x_0)(1 - 2^{-1/2}) \\ &\geqq 2^{-2}F_n(x_0) = 2^{-2}F_n(\mu_n)\cdot\theta\left(\frac{2x_0 - 2\alpha_n}{\beta_n - \alpha_n}\right) \\ &= 2^{-2}F_n(\mu_n)\cdot\theta\left(\frac{2\beta_n - 2x_0}{\beta_n - \alpha_n}\right) \geqq 2^{-2}F_n(\mu_n)\left(\frac{\beta_n - x_0}{\beta_n - \alpha_n}\right) \\ &\geqq 2^{-2}\left(\frac{\beta_n - \alpha_n}{2}\right)^{1/2}\left(\frac{\beta_n - x_0}{\beta_n - \alpha_n}\right) \geqq 2^{-5/2}\frac{(\beta_n - x_0)}{(\beta_n - \alpha_n)^{1/2}} \\ &\geqq 2^{(n-6)/2}\cdot(\beta_n - x_0). \end{aligned}$$

Consequently $\{\xi_n\}$ exists for which $x_0 < \xi_n = \mu_{2n}$ or β_{2n}, $(n=1, 2, 3, \cdots)$, with the result that

$$\left|\frac{F(\xi_n) - F(x_0)}{\xi_n - x_0}\right| \geqq 2^{n-3} \to \infty \quad \text{as} \quad n \to \infty$$

and the proof of the theorem will be complete as soon as we verify (5) and (6).

We observe that (5) consists of ten steps the reasons for which we now give in order. (i) This is a consequence of relation (2) of Part II and the fact that 4.7 and 4.6 imply $\mu_n \varepsilon H(F_n) = Z(F_{n+1}) = \mathfrak{Z}_{n+1}$. (ii) Arithmetic. (iii) In an alternating series of the type considered here the remainder term has the same sign as the first term omitted. Thus $(-1)^{n+1}F_{n+1}(x_0)$ and $R_n(x_0)$ are non-positive since n is even and positive. (iv) This is a consequence of 4.1(c) and $\theta(1)=1$. (v) From 3.4(a) it follows that $\theta(x) \leqq (x+3)/4$ for $0<x<1$. (vi) Arithmetic and the definition of μ_n. (vii) Inequality (4) of Part II. (viii) Arithmetic. (ix) Inequality (4) of Part II. (x) Arithmetic.

We treat (6) in a similar manner. (i) Relation (3) of Part II. (ii) An elementary inequality. (iii) Relation (1). (iv) Arithmetic. (v) This is a consequence of 4.1(c) and $\theta(1)=1$. (vi) From 3.3 we have $\theta(x)=\theta(2-x)$ for $0<x<2$. (vii) From 3.4(a) we have $x/2 \leqq \theta(x)$ for $0<x<1$. (viii) Inequality (4) of Part II. (ix) Arithmetic. (x) Inequality (4) of Part II.

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